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① triply graded link homology group $L \subset S^3$
Categorifying (colored) HOMFLY-PT polynomial
($SU(N)$ for all N simultaneously)

② refined open GW (BPS) invariants
for noncpt CY 3-fold $X = \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$

③ $U(1)$-instanton counting
= torus character of coord. ring of singular $U(1)$-instanton moduli sp.
sections of sheaves on

①②③ are related by "duality in physics"

$$① \xleftrightarrow[\text{large } N\text{-duality}]{} ② \xleftrightarrow[\text{geometric engineering}]{} ③$$

— I start with the case of $\begin{cases} \text{link} = \phi, & \text{cat'fied} \\ \text{original ver. (not refined)} \end{cases}$

① Chern–Simons partition function for S^3

Witten $Z(S^3; g) = \int_{\mathcal{A}(S^3)/\mathcal{G}} e^{2\pi i k CS(A)}$ $\quad q = e^{\frac{2\pi i}{k+N}}$
$G = SU(N)$

rigorous approach by Reshetikhin–Turaev
based on quantum groups

Now we make N : variable (Large N)
$Q := q^N$ new variable.

② GW invariant

"count" $\Sigma_g \xrightarrow[\text{holo. map}]{} X = \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$ resolved conifold

generating func.

$$F = \sum_{d \geq 1} \sum_{g \geq 0} g_s^{2g-2} Q^d \text{ "}\#\text{"} \{ f : \Sigma_g \to X \mid \deg = d \}$$

after $q = e^{i g_s}$

$$= \exp\left(- \sum_{d \geq 1} \frac{Q^d}{d (q^{d/2} - q^{-d/2})^2} \right) \underset{\text{up to pert.}}{=} \log Z \text{ in ②}$$

Faber – Pandharipande

③ $\sum_{n=0}^{\infty} \operatorname{ch} \mathbb{C}[S^n \mathbb{C}^2] \cdot Q^n$ $\mathbb{C}^* \curvearrowright \mathbb{C}^2$

$(x, y) \mapsto (qx, q^{-1}y)$

$\mathbb{C}[S^n \mathbb{C}^2] = S^n(\mathbb{C}[x,y])$

$\rightsquigarrow$ can be easily computed, $= \exp F$ above ②

$\operatorname{Hilb}^n \mathbb{C}^2 \to S^n \mathbb{C}^2$ resolution of singularities

$\therefore \mathbb{C}[S^n \mathbb{C}^2] \cong H^0(\operatorname{Hilb}^n \mathbb{C}^2, \mathcal{O})$

& $H^i(\operatorname{Hilb}^n \mathbb{C}^2, \mathcal{O}) = 0$ for $i > 0$.

localization formula (h : hook length)

$$\text{above} = \sum_{\lambda : \text{partition}} \frac{Q^{|\lambda|}}{\prod_{s \in \lambda} (1 - q^{h(s)})(1 - q^{-h(s)})}$$

Rem. integrality is not clear from this formula.

◎ refinement : add 1 more variable t

s.t. original $=$ refined $|_{t=-1}$

③ Consider torus action $(x,y) \mapsto (q_2 x, q_1^{-1} y)$

$$\exp\left(- \sum_{d=1}^{\infty} \frac{Q^d}{d(q_1^{\frac{d}{2}} - q_1^{-\frac{d}{2}})(q_2^{\frac{d}{2}} - q_2^{-\frac{d}{2}})} \right) \quad (t = -\sqrt{q_1/q_2})$$

② no rigorous definition so far.

physical (or naive) definition :

GW = Gopakumar-Vafa inv.

$\hookleftarrow$ Euler # of some moduli space (DT / PT)

replace euler number by Poincaré polynomial

Rem. Physists (Gukov-Schwarz-Vafa) said

this refinement has <u>no</u> deformation invariance

But $X = \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$ is rigid. So it is OK.

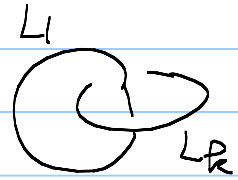
① no definition except for the final answer

(even physical) via ① = ② so far.....

but probably suggest ground ring $= S_Q^* \mathbb{C}[x,y]$

0 Link

($q = e^{\frac{2\pi i}{k+N}}$)

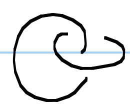
1 

Link in S^3 + $G = SU(N)$: cpt Lie group
$R_1, \cdots, R_k$: representations of G on each component of link

<u>Witten</u> $P_{G,R.}(L;q) = \dfrac{Z(S^3, L; q)}{Z(S^3; q)}$

where $Z(S^3, L; q) = \displaystyle\int_{\mathcal{A}(S^3)/\mathcal{G}} e^{2\pi i k CS(A)} \prod \mathrm{tr}_{R_i} \mathrm{Hol}_{L_i}(A)$

rigorous approach (in fact earlier) [RT]

e.g. $P_{SU(N),\square}$() $= 1 - q^{2N} + q^{2N} \cdot \dfrac{q^N - q^{-N}}{q - q^{-1}}$

- refinement (cat'tion)
  = triply graded link homology group
  (Khovanov-Rozansky)

The Poincaré poly. = poly in q, t, Q. $\xrightarrow[t=-1]{}$ link polynomial.

<u>Rem</u>. So far, the construction was given only in the case R = vector rep. (i.e. cat'tion of HOMFLY-PT poly.)

The definition is based on Hochschild cohomologies of certain module / $\mathbb{C}[x_1, \dots, x_m]$

basically <u>combinatorial</u>.

② open GW invariants (Ooguri-Vafa)

C_L = conormal bundle of $L \subset T^*S^3$

$$\begin{array}{ccccc} X & \xrightarrow[\text{resolution}]{} & \{xy+zw=0\} & \underset{\text{deformation}}{\rightsquigarrow} & \{xy+zw=\varepsilon\} = T^*S^3 \\ \cup & & \text{conifold} & & \cup \\ C_L' & & & & C_L \end{array}$$

C_L' ← lagrangian subvariety

which should be "canonical" deformation of C_L

(Taubes)

open GW inv.

= "count" $\Sigma \to X$ s.t. $\partial\Sigma \subset C_L'$

bdry condition is specified by partition

↔ representation of $SU(N)$

No rigorous definition so far.

even before refinement.

Li-Liu-Liu-Zhou : relative GW invariants

works at least for $L =$ Hopf link

moreover $\exists$ explicit answer (combinatorial) topological vertex

In fact, historically, topological vertex was introduced by [AMKV] via ① $\leftrightarrow$ ②

Then [LLLZ] gave the rigorous foundation.

Final answer :

$$Z\left(\,{}_{\mu}\;{}^{\nu}\,; q\right) = \sum_{\lambda} \frac{Q^{|\lambda|}\, s_{\mu^t}(q^{-\rho-\lambda^t})\, s_{\nu}(q^{-\rho-\lambda^t})}{\prod_{s\in\lambda}(1-q^{h(s)})(1-q^{-h(s)})}$$

s_μ : Schur function

$-\rho = (\frac{1}{2}, 1, \frac{3}{2}, 2, \cdots\cdots)$

refinement : Euler # $\Rightarrow$ Poincaré poly. ?

not clear at this moment.

But there is an advance by Pand.-Thomas

so in near future ?

③ (In this case, we give the refined version.)

Gukov et al :

T^2-char. of coord. ring of moduli of $U(1)$-instanton with singularities along axis

But for me, singular $U(1)$-instantons seem to be hard to consider. except for parabolic ones

$$\boxed{\begin{array}{l}\text{My proposal : Instead of singular instantons,} \\ \text{consider sheaves on } \mathrm{Hilb}^n\mathbb{C}^2 \text{, supported} \\ \text{on } L^n\mathbb{C} = \{ I \mid \mathrm{Supp}\, \mathcal{O}/I \subset x\text{-axis} \}\end{array}}$$

· Evidences

- case of Hopf link

$\mathcal{J}$: universal ideal sheaf for $\mathrm{Hilb}^n\mathbb{C}^2$

$$\mathcal{J}/x\text{-axis} \overset{\text{def.}}{=\!=} p_*(\mathcal{J} \otimes f^* \mathcal{O}_{x\text{-axis}})$$

$$\begin{tikzcd} & \mathrm{Hilb}^n\mathbb{C}^2 \times \mathbb{C}^2 \arrow[dl, "p"'] \arrow[dr, "f"] & \\ \mathrm{Hilb}^n\mathbb{C}^2 & & \mathbb{C}^2 \end{tikzcd}$$

From the above combinatorial expression

$$Z(\bigcirc\!\!\bigcirc ; q) = (\text{up to normalization})$$

$$= \sum_n Q^n \, \mathrm{ch}_{T^2} H^0(\mathrm{Hilb}^n\mathbb{C}^2, S_{\mu^t}(\mathcal{J}/x\text{-ax}) \otimes S_\nu(\mathcal{J}/x\text{-ax}))$$

S_ν : Schur functor

$S_\nu(J/x_{\alpha\alpha})$ is essentially the same as $S_\nu(\mathcal{O}^{[n]})$.

rk n v.b. over Hilb^n

(e.g. $\Lambda^i \mathcal{O}^{[n]}$
$S^i \mathcal{O}^{[n]}$ etc

<u>Conjecture</u> (Gukov et al, modified by N)
Poincaré poly. of triply graded link homology group for a link L
$$= \sum_{i,n} (-1)^i \mathrm{char}_{T^2} H^i(\mathrm{Hilb}^n \mathbb{C}^2, \mathcal{E}_L) \cdot Q^n$$
for some sheaf $\mathcal{E}_L$

Application to algebraic geometry

computation of $H^0(\mathrm{Hilb}^n \mathbb{C}^2, S_\lambda(\mathcal{O}^{[n]}) \otimes S_\nu(\mathcal{O}^{[n]}))$
— closely related to combinatorics of Macdonald polynomials.